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Short Note

Error localization in solution-adaptive grid methods

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In the article "On accuracy of adaptive grid methods for captured shocks" by Yamaleev and Carpenter [4], the performance of the adaptive methods on the simulation of smooth flow regions behind a shock wave is analyzed, and the results show that additional fine grids around a shock may not improve the accuracy downstream. The authors also comment that the accuracy in smooth solution region can be influenced by other factors, such as grid nonuniformity and the first-order accuracy of some shock-capturing schemes [1].

As a supplement to their work, this note emphasizes on the ability of the solution-adaptive grid methods to localize the numerical error of O(1) that appears in the neighborhood of captured shocks and other discontinuities. It is noticed that the choice of the error measure can be essential. A volume-weighted error measure is tested and compared with the classic error measure. The volume-weighted error measure is found to be proportional to the error in evaluating contour integrals. The ratio of the numbers of adaptive and uniform grid required for localizing the error within the same narrow region is further estimated by a simple analysis. It is shown that, for conducting one level of refinement, the number of cells for a locally adaptive grid increases approximately by a factor of 2, but it is 4 for the uniform grid.

The error measure that has been commonly used to evaluate the accuracy and convergence of solutions is

$$\|\phi - \phi^{\mathrm{ex}}\|_{L_K} = \left(\frac{\sum^N \left(\phi_i - \phi_i^{\mathrm{ex}}\right)^K}{N} \right)^{1/K}, \tag{1}$$

where ϕ_i and ϕ_i^{ex} are the numerical and exact values of ϕ at cell i, and N is the number of cells in the domain of interest. If $K = 2$ is used, error of the solution is measured in the L_2 sense. However, it is not appropriate for evaluating the solutions of adaptive methods when compared to uniform grids. The solution-adaptive techniques, in general, are to distribute grid cells to the regions, where solution errors are large, so that the large portion of total cells is distributed in the large-error regions. The problem of (1) may become devastating around shock waves, where peak numerical errors of O(1) hardly change with cell sizes (see, e.g., Figs. 1 and 7 of [4]) using a shock-capturing scheme. Solution-adaptive techniques simply distribute fine cells around shock waves, but these fine cells cannot decrease the numerical errors, although they do

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localize the errors in a narrower region. Error measure (1) emphasizes too much on the local large errors, and the ability of adaptive techniques to localize the numerical errors has not been properly taken into account.

We give here a simple numerical example to illustrate the ridicules performance of (1). Consider an isolated and stationary shock wave in a rectangular domain, we solve it using a coarse background grid as shown in Fig. 1(a), and compare two series of solutions using uniform and adaptive grids. Uniform grids with different grid sizes are obtained by repeatedly dividing the background grid cells, and adaptive grids obtained by dividing only the cells with large truncation errors following a vectorizable h-refinement strategy described in [3]. The conservation laws are solved by a second-order Godunov-type scheme with an approximate Riemann solver. An example of the four-level adaptive grid is shown in Fig. 1(b). Numerical errors in L_1 and L_2 senses for these two grids are plotted in Fig. 2. In this problem, suppose the adaptive technique covers the shock wave with the finest cells and treats well the abrupt change of grid interface, it is clear that the accuracy of solutions obtained on the adaptive grid is the same as that obtained on the corresponding uniform grid. For the two same solutions, errors measured on two grids, given by (1) both in L_1 and L_2 senses, show a contradictory behavior with decreasing grid size dx as seen from Fig. 2(a). The reason is that the uniform grid represents smooth regions with small errors using much more cells, or the adaptive grids distribute a larger portion of cells in the large-error region.

In order to represent the ability of error localization of solution-adaptive techniques, a volume-weighted error measure

$$\|\phi - \phi^{\mathrm{ex}}\|_{L_K} = \left(\frac{\sum^{N} \Omega_i \left(\phi_i - \phi_i^{\mathrm{ex}}\right)^K}{\sum^{N} \Omega_i} \right)^{1/K}, \tag{2}$$

where Ω_i is the volume of cell i, is tested. The results of the volume-weighted measure are also plotted in Fig. 2 for adaptive grids. For uniform grids, the volume-weighted measure degenerates to (1), so its results are not plotted in figure. It is seen that, using the volume-weighted error measure, the adaptive grid achieves very similar accuracy as its uniform grid counterpart.

Fig. 2(b) is a comparison of numerical errors as a function of total cell numbers for two grids. Interestingly enough one may derive two contradictory conclusions based on two error measures. Error measure (1) shows that the adaptive grid does not improve, or even worsen the accuracy compared to that calculated on a uniform grid with the same number of grid points. But if the volume-weighted error measure is

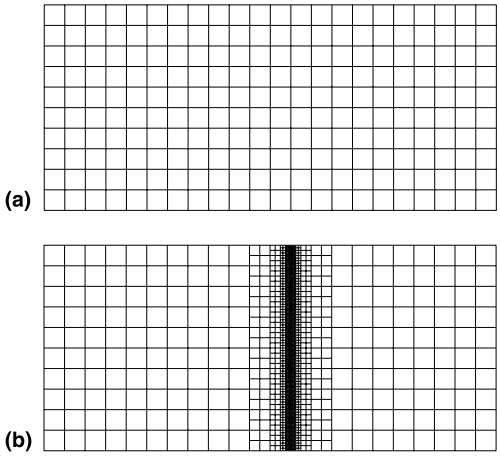

Fig. 1. Numerical grids: initial background grid (a) and four-level adaptive grid (b).

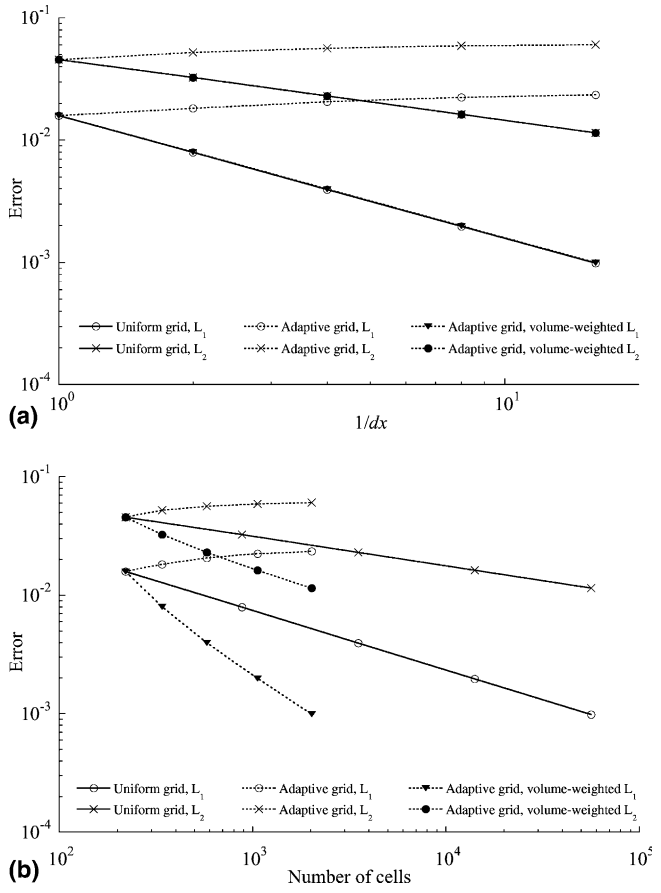

Fig. 2. Comparison of numerical errors of uniform and adaptive grids using different error measures.

used, the adaptive grid achieves the same accuracy as the uniform grid with an order of fewer magnitude cells.

In many practical applications, the volume-weighted measure is more reasonable to evaluate the accuracy of quantities that involves volume and contour integrals, such as the drag and lift coefficients, the total change of species in a domain. The volume-weighted measure is obviously good for quantities involving volume integrals. It is shown here that it also represents the accuracy of contour integrals. Consider an integral over a contour as sketched in Fig. 3, the error of the integrated quantity can be estimated as the summation of the numerical error in each cell along the contour

$$\mathrm{err} = \sum^{N_p} |\epsilon_i \Delta x_i|,$$

where Δx_i is the cell size and N_p is the total number of cells along the contour. One can divide all cells to a few groups according to their levels of refinement

$$\mathrm{err} = \sum^{N_0} |\epsilon_i \Delta x_0| + \sum^{N_1} |\epsilon_i \Delta x_1| + \cdots + \sum^{N_k} |\epsilon_i \Delta x_k| + \cdots,$$

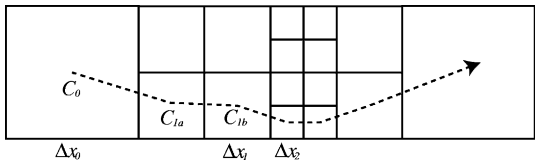

Fig. 3. An integral contour through locally refined cells.

where N_k are the number of kth-level cells along the contour, and their typical grid size is Δx_k, where k is the level of refinement. For an isotropic local refinement, one has $\Delta x_0 = 2^k \Delta x_k$. Then the error can be reformulated as

$$\mathrm{err} = \frac{1}{\Delta x_0}\left(\sum^{N_0}|\epsilon_i \Delta\Omega_0| + 2\sum^{N_1}|\epsilon_i \Delta\Omega_1| + \cdots + 2^k\sum^{N_k}|\epsilon_i \Delta\Omega_k| + \cdots\right),$$

where volume $\Omega_k = \Delta x_k \Delta x_k$. Noticing that numerical errors of the cells with the same level should be close, and there are approximately $2^k N_k$ cells of level k in a thread as sketched in Fig. 3, one gets an approximation

$$\mathrm{err} \approx \frac{1}{\Delta x_0}\left(\sum^{N_0}|\epsilon_i \Delta\Omega_0| + \sum^{2N_1}|\epsilon_i \Delta\Omega_1| + \cdots + \sum^{2^k N_k}|\epsilon_i \Delta\Omega_k| + \cdots\right) \approx \frac{1}{\Delta x_0}\sum^{N}|\epsilon_i \Delta\Omega_i|.$$

It is seen that the volume-averaged measure (2) in L_1 sense is proportional to the error of an integral over the contour that covers all threads.

The efficiency of the locally solution-adaptive method in localizing the numerical errors around discontinuities can be compared with that of the uniform grid by the following analysis. Consider a domain initially covered by N_0 coarsest cells, if one wants to restrict the error of O(1) around a discontinuity within a narrow region, 2^{-L} of the initial one, the uniform grid has to refine all initial cells, and thus requires

$$N^{\mathrm{uniform}} = 4^L N_0$$

cells. For the purpose of comparison, L is a positive integer, corresponding to the level of refinement for an adaptive grid. If a locally solution-adaptive grid is used, only a few rows of fine cells are necessary, as sketched in Fig. 4. One-level difference rule, the refinement levels of two neighboring cells different by no

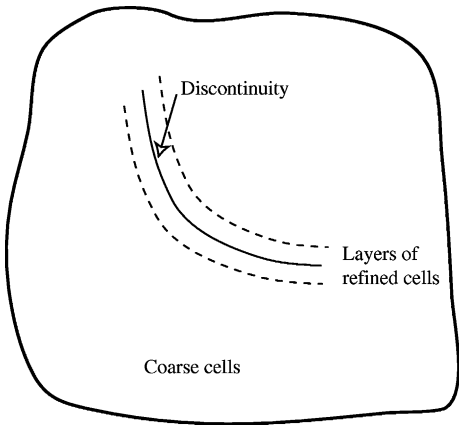

Fig. 4. Refining a discontinuity in locally solution-adaptive methods.

more than one, is followed here. Suppose the discontinuity is Γ long, and n' rows of fine cells is necessary, which is 8 in Fig. 1(b), one gets the number of kth-level cells, $(n'\Gamma)/(\Delta x_0/2^k)$. After the kth-level refinement, the $(k-1)$th-level cells is decreased by a factor of 1/4. So the net increase is $(n\Gamma)/(\Delta x_0/2^k)$, where $n = (3/4)n'$. The total number of cells can be obtained by summating all net increases

$$N^{\text{adaptive}} = N_0 + \sum_{k=1}^{L} \frac{n\Gamma}{\Delta x_0/2^k} = N_0 + \frac{n\Gamma}{\Delta x_0} \sum_{k=1}^{L} 2^k = N_0 + \frac{n\Gamma}{\Delta x_0} \left(2^{L+1} - 1\right).$$

Let $\alpha = (n\Gamma)/(\Delta x_0 N_0)$, one gets the ratio of cell numbers using adaptive and uniform grid approaches

$$\frac{N^{\text{adaptive}}}{N^{\text{uniform}}} = \frac{1-\alpha}{4^L} + \frac{2\alpha}{2^L}.$$

The parameter α represents the ratio of the number of refined initial cells to the total number, which is no more than 1. Even for $\alpha = 1$, one gets

$$\frac{N^{\text{adaptive}}}{N^{\text{uniform}}} = \frac{2}{2^L}. \tag{3}$$

A similar formula can be derived for three dimensions as well. Eq. (3) implies that for conducting one level of refinement or halving the grid size, the locally adaptive grid requires two times cells instead of four for the uniform grid. For a four-level refinement, the adaptive approach requires approximately one order of fewer magnitude cells compared with the uniform grid, which agrees well with the present result shown in Fig. 2(b).

Grid adaptation criterion for localizing numerical errors near discontinuities is different from the principle of error equidistribution. The principle is not valid for this purpose simply because it is impossible to distribute the error of O(1) using a nowadays shock-capturing scheme. The essence of adaptation criterion for localizing numerical errors is just to detect discontinuities in the computational domain. The criterion is actually a discontinuity-detecting or feature-detecting criterion. The adaptation criterion should be independent of the grid size since the numerical error near the discontinuities is nearly constant. In practice, one may choose the ratio of the second-order derivative term to the first-order one in the Taylor series of flow quantities as the adaptation criterion [2,3].

Present discussion on numerical accuracy, efficiency, and adaptation criterion is restricted for the numerical error near discontinuities. For smooth solution regions, other issues have to be taken into account, such as grid nonuniformity. The existence of first-order errors in the smooth solution region behind a shock wave [1,4] imposes an additional difficulty for the construction of a shock-capturing scheme and adaptation criterion.

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